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Pointing Accuracy of a Dual-Spin Satellite due to Torsional Appendage Vibrations

Mark C. Stabb* and A. L. Schlack†
University of Wisconsin, Madison, Wisconsin 53706

This paper deals with the attitude motion of a dual-spin satellite with a finite sized rigid body attached to the end of a flexible beam. The equations of motion are derived using Lagrange's equations and are solved using the perturbation technique known as the Krylov-Bogoliubov-Mitropolsky method. The special case of torsional flexibility is presented in its entirety. The relationships between the satellite, beam and tip mass parameters, and pointing accuracy of the satellite are examined.

Introduction

THE dynamics of satellites with moving internal and external parts have been investigated with some diligence since shortly after the first artificial satellite was launched over 30 years ago.[1] This is a result of the fact that many satellites must be pointed accurately to achieve mission goals, and so an understanding of their motion is crucial for engineers who must design and control the craft. In many cases the equations of motion for the spacecraft are numerically integrated for a particular model configuration of the spacecraft to explain or predict the behavior of an actual craft. In problems in which an analytical solution is desired, a typical approach is to linearize the equations of motion before solving (for example, see Kulla[2]). In both the cases of numerical integration and linearizing the equations, fundamental information that the nonlinearities give about the stability and attitude of the problem is lost.

In a few cases an attempt is made to overcome this problem by applying a perturbation technique to the equations of motion to study the full-field parameter space of the problem. Typical methods employed are the method of multiple scales, a first-order averaging method like the Krylov-Bogoliubov technique,[3] or generalized averaging that is sometimes called improved first-order averaging. Suleman and Modi[4] applied the Krylov-Bogoliubov technique to a flexible but linearized system and found that "the analytical solution predicted the response with a surprising degree of accuracy...." An extension to this technique, called the Krylov-Bogoliubov-Mitropolsky (KBM) method, has been used by some authors, but these studies have been limited to the investigation of appendage vibration, not to the overall system motion and the related pointing accuracy (for example, see Kammer and Schlack[5]). The KBM method is used in the present work.

To first order, multiple scales and averaging give the same results.[6] An improvement on first-order averaging can be obtained by using either generalized averaging of the KBM method. Both of these methods yield not only the same long-term variations given by regular averaging or multiple scales but also give short-term fluctuations. Generalized averaging requires that the equations be in periodic standard form[6]; KBM has no such requirement. Therefore, this paper uses the KBM method instead of generalized averaging since the equations of motion obtained for the satellite are not in periodic standard form.

By deriving an approximate analytical solution of the nonlinear equations of motion, the work of this paper allows examination of the effects of system parameters on the motion of the spacecraft in question. Although resonances are easily obtained from a simple examination of the fundamental frequencies of the problem, this work allows the examination of behavior away from resonance. This provides the opportunity to look for local minima of the perturbation from rigid-body behavior as a function of the system parameters and hence allows for a more optimally designed spacecraft. In addition, control of the spacecraft could be more efficiently implemented if a general idea of the form of the motion is available. With numerically computed results, it is not possible to examine the effects of individual parameters without many repeated evaluations of the motion. The approach in this paper provides full-field parametric relationships for the examination of all system parameters.

Also of note is the fact that the effects of finite size tip masses are investigated. Rotary inertia of the tip mass introduces a set of equations of motion for the tip mass that are coupled to the equations of motion for the system through the dynamics of the elastic beam. Very little has been done concerning analytical examination of such a tip mass. These results will provide a new tool for the design and control of spacecraft of this type, as well as introducing the KBM approach to the body of literature relating to this topic.

Problem Formulation

Model Description

The model to be examined is depicted in Fig. 1. It is composed of a main body or platform, on which is mounted a rotor that spins relative to the platform. In addition to the rotor, a number of appendages are cantilevered off the main body. A general formulation would allow for an arbitrary number of appendages, but Fig. 1 and the example to be examined have only one. Extending this approach to more than one appendage is easily accomplished but does little to help demonstrate the method applied. The points c_i are the centers of mass for each of the components, and c_m is the center of mass of the system. The rotor is considered part of the main body for the purpose of determining center of mass. Figure 2 shows the body in a deformed state, with tip rotation about the beam axis. This is the only displacement mode to be considered, as might happen if the appendage were constrained by guy wires.[7]

The following assumptions are made in this model:

1) A set of coordinate axes is attached to the center of mass of the platform and is parallel to the principal axes in the undeformed state. The center of mass of the tip mass and hence the system lies on the y axis.

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*Graduate Student, Department of Engineering Mechanics and Astronautics, 1415 Johnson Drive. Student Member AIAA.

†Professor, Department of Engineering Mechanics and Astronautics, 1415 Johnson Drive. Member AIAA.

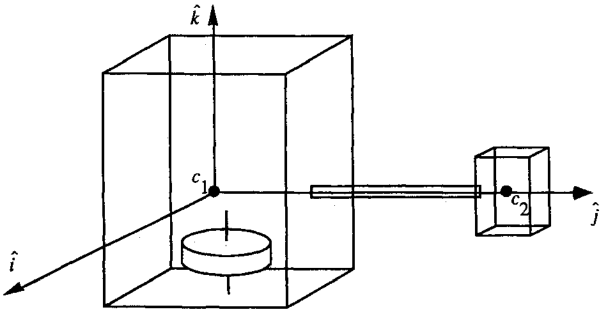

Fig. 1 Dual-spin satellite with appendage.

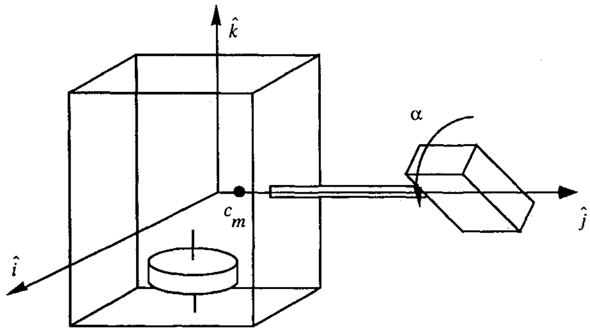

Fig. 2 Satellite in the deformed configuration.

2) In the undeformed state, the principal moments of inertia of the system about the x and y axes are equal, the moment of inertia about the z axis is arbitrary, and the tip mass has its principal axes aligned with the system's axes.

3) A symmetric, rigid rotor is aligned with the z axis.

4) The system is force and torque free.

5) The beam is massless and has a circular cross section.

6) The initial values of the platform's angular velocity components are "small" (in a sense to be defined later) compared with that of the rotor and the natural frequency of the beam.

7) The tip mass is assumed "small" compared with the spacecraft as a whole.

Equations of Motion

The equations of motion are formulated by creating the Lagrangian for the total system and applying the extended Hamilton's principle along with Lagrange's equations for quasicoordinates.[8] Let $\theta(y,t)$ be the rotation of the beam relative to the xyz axes and assume that the rotation of the continuous beam can be replaced by a series of generalized coordinates in time and admissible functions in space, such as

$$\theta(y,t) = \sum_{i=1}^{n} \theta_i(t) Y_{\theta i}(y) \tag{1}$$

so the equations can be converted into ordinary differential equations in time only. The rotation that the tip mass experiences is simply Eq. (1) evaluated at $y = L$ where L is the length of the beam. The kinetic energy of the platform and rotor is given by

$$T_1 = \tfrac{1}{2}\omega^T[I_1]\omega + \tfrac{1}{2}I_r(\omega_s + \omega_z)^2 + \tfrac{1}{2}m_1\dot{r}_1 \cdot \dot{r}_1 \tag{2}$$

where $\omega = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$ is the angular velocity of the platform, ω_s is the angular velocity of the rotor relative to the platform, I_r is the moment of inertia of the rotor about its spin axis, and $[I_1]$ is the inertia tensor of the platform and rotor together about c_1 except about the z axis where it is just for the platform. The velocity of point c_1 is $\dot{r}_1 = \omega \times r_1$ where r_1 is the vector from c_m to c_1. The rotor's energy from spin about the z axis is given by the second term in Eq. (2). The kinetic energy of the tip mass is given by

$$T_2 = \tfrac{1}{2}\omega_2^T[I_2]\omega_2 + \tfrac{1}{2}m_2\dot{r}_2 \cdot \dot{r}_2 \tag{3}$$

where

$$\omega_2 = (\omega_x \cos\theta - \omega_z \sin\theta)\hat{\imath} + [\omega_y + \dot{\theta}(L,t)]\hat{\jmath} + (\omega_z \cos\theta + \omega_x \sin\theta)\hat{k}$$

is the angular velocity of the tip mass expressed in terms of principal coordinates of the tip mass, and $[I_2] = [I_x, I_y, I_z]$ is the principal inertia tensor of the tip mass about c_2. It is also diagonal relative to the xyz frame in the undeformed case. The velocity of point c_2 is $\dot{r}_2 = \omega \times r_2$ where r_2 is the vector from c_m to c_2. Using the parallel axis theorem, it can be shown that the total kinetic energy of the satellite reduces to

$$T = \tfrac{1}{2}\omega^T\big([I_s] - [I_2]\big)\omega + \tfrac{1}{2}I_r(\omega_s + \omega_z)^2 + \tfrac{1}{2}\omega_2^T[I_2]\omega_2 \tag{4}$$

where $[I_s] = [A, A, C]$ is the moment of inertia tensor for the entire satellite not including the rotor about its spin axis.

The potential energy of the system comes from the elastic beam and is given by

$$V = \frac{1}{2}\int_0^L \left(\frac{\partial\theta}{\partial y}\right)^2 JG \, dy \tag{5}$$

For the assumption of a massless beam, the use of only one admissible function y/L gives exact results. At the tip mass we define $\alpha(t) = \theta(L,t)$. This gives $\theta(y,t) = \alpha(t)y/L$. With this simplification and the introduction of the beam stiffness $K_y = JG/L$, the potential energy can be integrated and simply becomes

$$V = \frac{1}{2}\int_0^L \left[\frac{\alpha(t)}{L}\right]^2 JG \, dy = \frac{JG\alpha(t)^2}{2L} = \frac{K_y\alpha(t)^2}{2} \tag{6}$$

With the energy now only a function of time, Hamilton's principle reduces to Lagrange's equations. Since the strain energy V is accounted for in the Lagrangian, the generalized forces Q_i are solely from forces not derivable from a potential function. Such effects as gravity gradient torques, thermal gradients, and atmospheric drag can all be included here. For this analysis, the various Q_i are taken as zero except for the torque on the rotor, which will be allowed to be nonzero.

The Lagrangian for the system $T - V$ is used in the formulation of the equations of motion using Lagrange's equations. The equations of motion obtained for the system are

$$\begin{aligned}
&\left[A - (I_x - I_z)\sin^2\alpha\right]\dot{\omega}_x + \left[C - A + (I_x - I_z)\sin^2\alpha\right]\omega_y\omega_z \\
&\quad - (I_x - I_z)\cos\alpha\sin\alpha(\omega_x\omega_y + \dot{\omega}_z) \\
&\quad - 2(I_x - I_z)\cos\alpha\sin\alpha\,\omega_x\dot{\alpha} \\
&\quad + \left[-I_y + (I_x - I_z)(\sin^2\alpha - \cos^2\alpha)\right]\omega_z\dot{\alpha} \\
&\quad + I_r\omega_y(\omega_z + \omega_s) = 0
\end{aligned} \tag{7}$$

$$\begin{aligned}
&A\dot{\omega}_y + \left[A - C - 2(I_x - I_z)\sin^2\alpha\right]\omega_x\omega_z \\
&\quad + (I_x - I_z)\cos\alpha\sin\alpha(\omega_x^2 - \omega_z^2) + I_y\ddot{\alpha} \\
&\quad - I_r\omega_x(\omega_z + \omega_s) = 0
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\left[C + (I_x - I_z)\sin^2\alpha\right]\dot{\omega}_z + I_r(\dot{\omega}_z + \dot{\omega}_s) + (I_x - I_z)\sin^2\alpha\,\omega_x\omega_y \\
&\quad + (I_x - I_z)\cos\alpha\sin\alpha(\omega_y\omega_z - \dot{\omega}_x) \\
&\quad - 2(I_x - I_z)\cos\alpha\sin\alpha\,\omega_z\dot{\alpha} \\
&\quad + \left[I_y + (I_x - I_z)(\sin^2\alpha - \cos^2\alpha)\right]\omega_x\dot{\alpha} = 0
\end{aligned} \tag{9}$$

$$I_r(\dot{\omega}_s + \dot{\omega}_z) = \tilde{T} \tag{10}$$

$$I_y\ddot{\alpha} + I_y\dot{\omega}_y + K_y\alpha + (I_x - I_z)(\omega_x \cos\alpha - \omega_z \sin\alpha)$$

$$\times (\omega_x \sin\alpha + \omega_z \cos\alpha) = 0 \tag{11}$$

where $\tilde{T}$ is the torque on the rotor from the drive motor.

To apply a perturbation method to the problem, the equations are nondimensionalized and a quantity ϵ that is small compared with unity is introduced. In this way the assumptions of small platform angular velocity and small relative mass of the tip mass are explicitly taken into account. The equations of motion are then rearranged and expanded in powers of ϵ, keeping only terms to first order in ϵ.

First, the unperturbed [$\mathcal{O}(\epsilon^0)$] equations are solved, and the result is a solution for α in terms of cosines and sines as functions of time. When the solution of the unperturbed equations is substituted into the first-order equations in ϵ, a set of forced first- and second-order differential equations is obtained. The forcing terms of the form $\cos(\alpha)$ become terms of the form $\cos\{\cos[f(t)]\}$. Since the particular solution of the differential equation due to the forcing is required, we replace the $\cos(\alpha)$ and $\sin(\alpha)$ terms in the equations with their power series to simplify the computation of the first-order solution. For example,

$$\cos(\alpha) \to 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24} - \cdots \tag{12}$$

Our problem term then becomes

$$\cos\left\{\cos\left[f(t)\right]\right\} \to 1 - \frac{\cos^2\left[f(t)\right]}{2} + \frac{\cos^4\left[f(t)\right]}{24} - \cdots \tag{13}$$

which can now be handled in a straightforward manner through the use of trigonometric identities. If α is not too large, the power series can be truncated after only a few terms with a high degree of accuracy. In the case presented here, terms to α^2 are retained to exhibit some of the nonlinear flavor of the problem while keeping the required algebra to a minimum.

We now assume that the rotor spins freely or that the motor torque exactly cancels friction so $\tilde{T} = 0$. The assumption that $\dot{\omega}_s = 0$ yields similar results. With the following definitions, the nondimensional equations of motion can be generated:

$$\bar{I}_x = I_x/(A\epsilon) \qquad \bar{I}_y = I_y/(A\epsilon) \qquad \bar{I}_z = I_z/(A\epsilon)$$

$$\bar{I}_r = I_r/A \qquad \bar{C} = C/A \qquad \bar{\alpha} = \alpha$$

$$\bar{\omega}_s = \omega_s/\omega_z' \qquad \omega_z' = (\omega_s + \omega_z)I_r/A \qquad \bar{K}_y = K_y/(\omega_z'^2 A\epsilon)$$

$$\bar{\omega}_x = \omega_x/(\omega_z'\epsilon) \qquad \bar{\omega}_y = \omega_y/(\omega_z'\epsilon) \qquad \bar{\omega}_z = \omega_z/(\omega_z'\epsilon)$$

$$\gamma^2 = \bar{K}_y/\bar{I}_y \qquad \tau = \omega_z' t \qquad \mathrm{d}/\mathrm{d}t = \omega_z'\,\mathrm{d}/\mathrm{d}\tau \tag{14}$$

where ω_z' is the precession frequency of the spacecraft with no flexibility, τ is nondimensional time, and γ is the nondimensionalized natural frequency of the beam. In addition to τ and γ, quantities with an overbar are now nondimensional as well. Note that the beam stiffness is assumed to be $\mathcal{O}(\epsilon^1)$ here. This is due to the fact that $\bar{I}_y$ is $\mathcal{O}(\epsilon^1)$ and the natural frequency of the beam and tip mass together is assumed to be of $\mathcal{O}(\epsilon^0)$. Using Eq. (14), Eqs. (7-11) now become

$$\bar{\omega}_x' + \bar{\omega}_y + \epsilon\left[(\bar{C}-1)\bar{\omega}_y\bar{\omega}_z - (\bar{I}_x - \bar{I}_z)(2\bar{\omega}_x\bar{\alpha}\bar{\alpha}'\right.$$

$$\left. + \bar{\omega}_z\bar{\alpha}' + \bar{\omega}_z'\bar{\alpha} + \bar{\alpha}^2\bar{\omega}_x') - \bar{I}_y\bar{\omega}_z\bar{\alpha}'\right] = 0 \tag{15}$$

$$\bar{\omega}_y' - \bar{\omega}_x + \bar{I}_y\bar{\alpha}'' + \epsilon(1-\bar{C})\bar{\omega}_x\bar{\omega}_z = 0 \tag{16}$$

$$\bar{C}\bar{\omega}_z' + \bar{I}_r(1/\epsilon\bar{\omega}_s' + \bar{\omega}_z')$$

$$+ \epsilon\left[(\bar{I}_x - \bar{I}_z)(2\bar{\omega}_z\bar{\alpha}\bar{\alpha}' - \bar{\omega}_x\bar{\alpha}' - \bar{\omega}_x'\bar{\alpha} + \bar{\omega}_z'\bar{\alpha}^2) + \bar{I}_y\bar{\omega}_x\bar{\alpha}'\right] = 0 \tag{17}$$

$$\bar{I}_r(\bar{\omega}_s' + \epsilon\bar{\omega}_z') = 0 \tag{18}$$

$$\bar{\alpha}'' + \gamma^2\bar{\alpha} + \epsilon\bar{\omega}_y' = 0 \tag{19}$$

where primes $(\ )'$ denote differentiation with respect to τ.

Solution of Equations

Equation (18) is the complete equation for the rotor; there are no higher order terms that have been ignored. Given this, we see that it is immediately integrable, yielding

$$\bar{\omega}_r = \bar{\omega}_s + \epsilon\bar{\omega}_z = \text{const} \tag{20}$$

where $\bar{\omega}_r$ is the nondimensional angular velocity of the rotor with respect to inertial space. Equation (20) can be substituted into Eq. (17) to simplify that equation to

$$\bar{C}\bar{\omega}_z' + \epsilon\left[(\bar{I}_x - \bar{I}_z)(2\bar{\omega}_z\bar{\alpha}\bar{\alpha}' - \bar{\omega}_x\bar{\alpha}' - \bar{\omega}_x'\bar{\alpha} + \bar{\omega}_z'\bar{\alpha}^2) + \bar{I}_y\bar{\omega}_x\bar{\alpha}'\right] = 0 \tag{21}$$

To get an approximate solution to this set of coupled nonlinear equations, the KBM method is applied. It is typically applied to a single second-order differential equation[8-10]; here however, since the unperturbed solutions for $\bar{\omega}_x$, $\bar{\omega}_y$, and $\bar{\alpha}$ behave like coupled harmonic oscillators, the method is extended to the system of coupled first- and second-order equations that describe the satellite's motion.[11]

The first step is to solve the system for the unperturbed case. This corresponds to a rigid, axisymmetric, dual-spin satellite, and the solution is easily obtained as found in Rimrott.[12] The unperturbed solution is

$$\bar{\omega}_x = a\cos(\omega_z' t + \beta_x) + f_x\cos(\gamma t + \beta_q) \tag{22}$$

$$\bar{\omega}_y = a\sin(\omega_z' t + \beta_x) + f_y\sin(\gamma t + \beta_q) \tag{23}$$

$$\bar{\omega}_z = b \tag{24}$$

$$\bar{\alpha} = e\cos(\gamma t + \beta_q) \tag{25}$$

where

$$f_x = \frac{e\gamma^2\bar{I}_y}{\gamma^2 - 1} \qquad \text{and} \qquad f_y = \frac{e\gamma^3\bar{I}_y}{\gamma^2 - 1} \tag{26}$$

Expanding on the KBM approach, we propose a solution of the form

$$\bar{\omega}_x = a\cos(\xi) + f_x\cos(\psi) + \epsilon\Omega_x(a,b,e,\xi,\psi) \tag{27}$$

$$\bar{\omega}_y = a\sin(\xi) + f_y\sin(\psi) + \epsilon\Omega_y(a,b,e,\xi,\psi) \tag{28}$$

$$\bar{\omega}_z = b + \epsilon\Omega_z(a,b,e,\xi,\psi) \tag{29}$$

$$\bar{\alpha} = e\cos(\psi) + \epsilon\tilde{A}(a,b,e,\xi,\psi) \tag{30}$$

In the unperturbed case, the arguments of the sines and cosines would simply be

$$\xi = \tau + \beta_x \qquad \text{and} \qquad \psi = \gamma\tau + \beta_q \tag{31}$$

Now they are allowed to vary slightly from their unperturbed values and have the properties

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = 1 + \epsilon\xi_1, \qquad \xi(\tau = 0) = \beta_x \tag{32}$$

$$\frac{d\psi}{dt} = \gamma + \epsilon\psi_1, \qquad \psi(\tau=0) = \beta_q \tag{33}$$

The coefficients that were formerly constants are now allowed to vary with time to first order in our small parameter ϵ:

$$\frac{da}{dt} = \epsilon A_1, \qquad \frac{db}{dt} = \epsilon B_1, \qquad \frac{de}{dt} = \epsilon E_1 \tag{34}$$

When the KBM method is applied to the problem,[11] the following results are obtained:

$$A_1 = B_1 = E_1 = 0 \tag{35}$$

$$\xi_1 = b(\bar{C}-1) + \frac{e^2}{4}(\bar{I}_x - \bar{I}_z) - \frac{\bar{I}_y}{2(\gamma^2-1)} \tag{36}$$

$$\psi_1 = f_y/(2e) \tag{37}$$

The long-term variations in Eqs. (35–37) come from the fact that the solutions must not grow in an unbounded fashion as functions of time. In other words, resonances in the forcing terms must be avoided, giving the constraints needed to determine these coefficients.

The second part of the solution involves finding the short-term variations Ω_x, Ω_y, Ω_z, and $\tilde{A}$. Recall that when the solution to the unperturbed equations is substituted into the first-order equations, we obtain a set of forced first- and second-order differential equations. With the power series expansion of $\cos(\alpha)$ and $\sin(\alpha)$ mentioned earlier, the forcing becomes the sum of sines and cosines of the natural frequencies of the problem. The short-term variations are simply the particular solution to these equations with the added constraint that the initial values of the short-term variations be zero:

$$\Omega_x = \Omega_{x0}\sin(\xi) + \Omega_{x1}\cos(\xi) + \Omega_{x2}\cos(\psi) + \Omega_{x3}\cos(3\psi) + \Omega_{x4}\cos(\xi+2\psi) + \Omega_{x5}\cos(\xi-2\psi) \tag{38}$$

$$\Omega_y = \Omega_{y0}\cos(\xi) + \Omega_{y1}\sin(\xi) + \Omega_{y2}\sin(\psi) + \Omega_{y3}\sin(3\psi) + \Omega_{y4}\sin(\xi+2\psi) + \Omega_{y5}\sin(\xi-2\psi) \tag{39}$$

$$\Omega_z = \Omega_{z0} + \Omega_{z1}\cos(2\psi) + \Omega_{z2}\cos(\xi-\psi) + \Omega_{z3}\cos(\xi+\psi) \tag{40}$$

$$\tilde{A} = \tilde{A}_0\cos(\psi) + \tilde{A}_1\sin(\psi) + \tilde{A}_2\cos(\xi) \tag{41}$$

The coefficient for each of the terms in Ω_z and $\tilde{A}$ can be found individually, giving

$$\Omega_{z1} = -\frac{e}{4\bar{C}\gamma}\left[2\gamma(be-f_x)(\bar{I}_x-\bar{I}_z) + \gamma f_x \bar{I}_y\right] \tag{42}$$

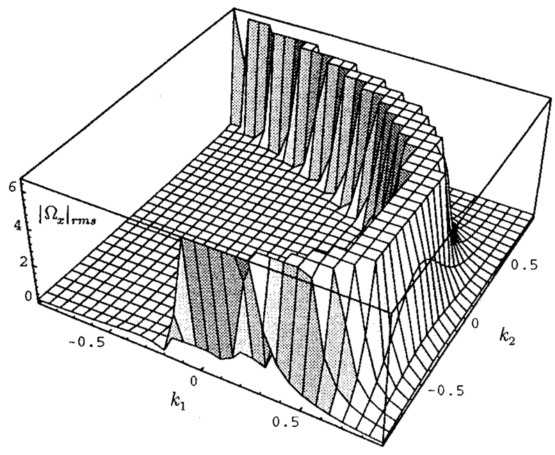

Fig. 3 Perturbation amplitude as a function of tip mass shape.

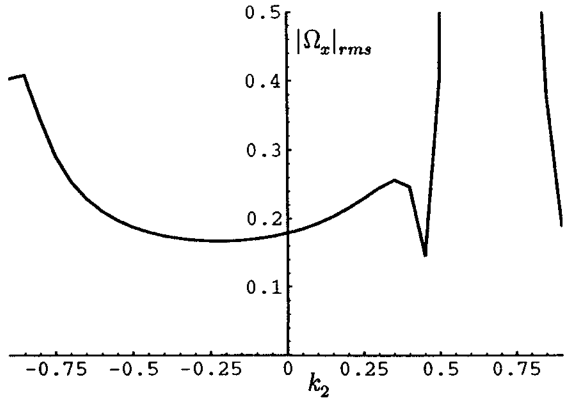

Fig. 4 Perturbation amplitude along the line $k_1 = 0$.

$$\Omega_{z2} = \frac{ae}{2\bar{C}(\gamma-1)}\left[(\gamma-1)(\bar{I}_x-\bar{I}_z) - \gamma\bar{I}_y\right] \tag{43}$$

$$\Omega_{z3} = \frac{ae}{2\bar{C}(\gamma+1)}\left[(\gamma+1)(\bar{I}_x-\bar{I}_z) - \gamma\bar{I}_y\right] \tag{44}$$

$$\Omega_{z0} = -\left[\Omega_{z1}\cos(2\beta_q) + \Omega_{z2}\cos(\beta_x-\beta_q) + \Omega_{z3}\cos(\beta_x+\beta_q)\right] \tag{45}$$

and

$$\tilde{A}_2 = -\frac{a}{\gamma^2-1} \tag{46}$$

$$\tilde{A}_0 = -\tilde{A}_2\left[\cos(\beta_x)\cos(\beta_q) + (1/\gamma)\sin(\beta_x)\sin(\beta_q)\right] \tag{47}$$

$$A_1 = \tilde{A}_2\left[-\cos(\beta_x)\sin(\beta_q) + (1/\gamma)\sin(\beta_x)\cos(\beta_q)\right] \tag{48}$$

The solutions for the Ω_x and Ω_y coefficients require that they be solved in pairs since Eqs. (15) and (16) are coupled. Again, the initial condition constraints given earlier yield the final equations to solve for all of the coefficients. We have

$$\Omega_{x2} = -\frac{e\gamma^2}{4(\gamma^2-1)^3}\left[4\bar{I}_y^2\gamma^2 + 4b\bar{I}_y(\gamma^2-1)(\gamma^2+1)(1-\bar{C}) - 3e^2\gamma^2\bar{I}_y(\gamma^2-1)(\bar{I}_x-\bar{I}_z) - 4b(\gamma^2-1)^2(\bar{I}_x+\bar{I}_y-\bar{I}_z)\right] \tag{49}$$

$$\Omega_{y2} = -\frac{e\gamma}{4(\gamma^2-1)^3}\left[2\bar{I}_y^2\gamma^4(3-\gamma^2) + 8b\bar{I}_y(\gamma^2-1)(1-\bar{C}) - 3e^2\gamma^2\bar{I}_y(\gamma^2-1)(\bar{I}_x-\bar{I}_z) - 4b(\gamma^2-1)^2(\bar{I}_x+\bar{I}_y-\bar{I}_z)\right] \tag{50}$$

$$\Omega_{x3} = \frac{9\gamma^4 e^3\bar{I}_y(\bar{I}_x-\bar{I}_z)}{4(\gamma^2-1)(9\gamma^2-1)} \tag{51}$$

$$\Omega_{y3} = \frac{3\gamma^3 e^3\bar{I}_y(\bar{I}_x-\bar{I}_z)}{4(\gamma^2-1)(9\gamma^2-1)} \tag{52}$$

$$\Omega_{x4} = \frac{(1+2\gamma)^2}{16\gamma(1+\gamma)}ae^2(\bar{I}_x-\bar{I}_z) \tag{53}$$

$$\Omega_{y4} = \frac{1+2\gamma}{16\gamma(1+\gamma)}ae^2(\bar{I}_x-\bar{I}_z) \tag{54}$$

$$\Omega_{x5} = \frac{(1-2\gamma)^2}{16\gamma(1-\gamma)}ae^2(\bar{I}_x-\bar{I}_z) \tag{55}$$

$$\Omega_{y5} = \frac{1-2\gamma}{16\gamma(1-\gamma)}ae^2(\bar{I}_x-\bar{I}_z) \tag{56}$$

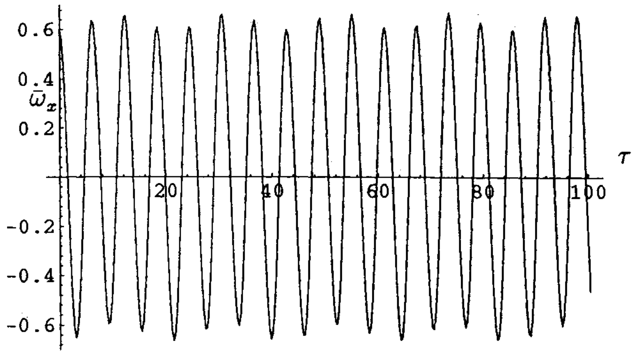

Fig. 5 $\bar{\omega}_x$ **as a function of** τ.

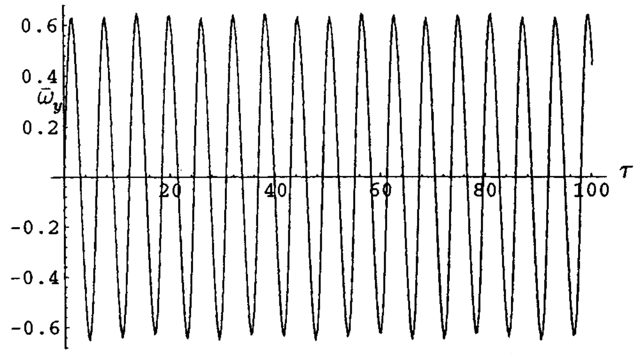

Fig. 6 $\bar{\omega}_y$ **as a function of** τ.

$$\begin{aligned}\Omega_{x0} = \Bigg\{ & -\left[a\,\frac{(\bar{I}_x-\bar{I}_z)(\gamma^2-1)e^2-2\bar{I}_y}{2}\right]\sin(2\beta_x) \\ & +\left[2(\gamma^2-1)(\Omega_{x4}-\Omega_{x5}+\Omega_{y4}-\Omega_{y5})\right]\sin(2\beta_q) \\ & +\left[2(\gamma^2-1)(\Omega_{y2}-\Omega_{x2})\right]\sin(\beta_x+\beta_q) \\ & -\left[2(\gamma^2-1)(\Omega_{y2}+\Omega_{x2})\right]\sin(\beta_x-\beta_q) \\ & +\left[2(\gamma^2-1)(\Omega_{y3}-\Omega_{x3})\right]\sin(\beta_x+3\beta_q) \\ & -\left[2(\gamma^2-1)(\Omega_{y3}+\Omega_{x3})\right]\sin(\beta_x-3\beta_q) \\ & +\left[2(\gamma^2-1)(\Omega_{y4}-\Omega_{x4})\right]\sin(2\beta_x+2\beta_q) \\ & +\left[2(\gamma^2-1)(\Omega_{x5}-\Omega_{y5})\right]\sin(2\beta_x-2\beta_q)\Bigg\}\Big/\left[4(\gamma^2-1)\right] \end{aligned} \tag{57}$$

$$\begin{aligned}\Omega_{x1} = \Bigg\{ & \left[a\,\frac{(\bar{I}_x-\bar{I}_z)(\gamma^2-1)e^2-2\bar{I}_y}{2}\right] \\ & +\left[-a\,\frac{(\bar{I}_x-\bar{I}_z)(\gamma^2-1)e^2-2\bar{I}_y}{2}\right]\cos(2\beta_x) \\ & -\left[2(\gamma^2-1)(\Omega_{x4}+\Omega_{x5}+\Omega_{y4}+\Omega_{y5})\right]\cos(2\beta_q) \\ & +\left[2(\gamma^2-1)(\Omega_{y2}-\Omega_{x2})\right]\cos(\beta_x+\beta_q) \\ & -\left[2(\gamma^2-1)(\Omega_{y2}+\Omega_{x2})\right]\cos(\beta_x-\beta_q) \\ & +\left[2(\gamma^2-1)(\Omega_{y3}-\Omega_{x3})\right]\cos(\beta_x+3\beta_q) \\ & -\left[2(\gamma^2-1)(\Omega_{y3}+\Omega_{x3})\right]\cos(\beta_x-3\beta_q) \\ & +\left[2(\gamma^2-1)(\Omega_{y4}-\Omega_{x4})\right]\cos(2\beta_x+2\beta_q) \\ & +\left[2(\gamma^2-1)(\Omega_{y5}-\Omega_{x5})\right]\cos(2\beta_x-2\beta_q)\Bigg\}\Big/\left[4(\gamma^2-1)\right] \end{aligned} \tag{58}$$

$$\Omega_{y0} = -\Omega_{x0} \tag{59}$$

$$\Omega_{y1} = \Omega_{x1} - a\left[e^2(\bar{I}_x-\bar{I}_z)+2\bar{I}_y/(\gamma^2-1)\right]\Big/4 \tag{60}$$

Once these coefficients are computed, the equations can be evaluated directly. Therefore, although the initial calculations may be complex, a solution of this form could be found for a very general system with a large number of appendages and system parameters. Then specific configurations (i.e., only one appendage with tip mass) could be obtained by setting unneeded parameters to zero. Thus, this method need only be applied once and then evaluated as needed. An advantage of this is that it does not require the computationally expensive procedure of numerical integration but only the re-evaluation of the coefficients each time the values of the parameters are changed.

Results

A number of interesting results are obtained. We see that the amplitudes of the perturbations do not vary with time. This is a reasonable result in light of the assumption of no damping; the energy is a function of amplitudes, and there is no mechanism to dissipate the energy of the system. However, the frequencies of the unperturbed equations are altered. This is consistent with the introduction of flexibility and coupling to the problem. In general, a flexible problem does not have the same natural frequencies as the corresponding rigid uncoupled problem.

One application of determining the coefficients for every term in each of the short-term variations is to look at the total expected perturbation. A reasonable measure of the total perturbation on one of the variables of the problem is the root mean square of the coefficients for the associated short-term variation. In general, most of the frequencies found in the various parts of each short-term variation will not appear as integer ratios of one another (in fact some are prohibited due

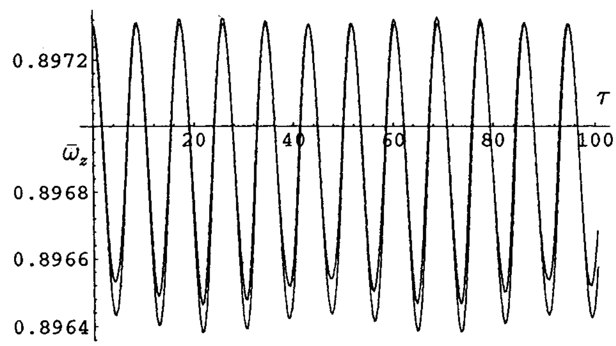

Fig. 7 $\bar{\omega}_z$ **as a function of** τ.

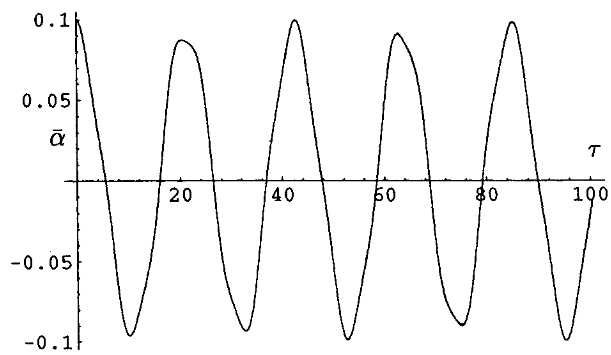

Fig. 8 $\bar{\alpha}$ **as a function of** τ.

to resonance considerations), so this seems a logical way to get the "average" value to the perturbation. In addition since the energy of the system depends on the square of the beam rotation and the square of the angular frequencies of the platform, the root mean square also gives some idea of the energy of the perturbation, which is important in determining the control effort required for stabilization. In this manner, the size of the perturbations can be examined as functions of the parameters of the system. Since there are too many parameters to graph in a simple form, we now consider the case where the only two free parameters are

$$k_1 = \frac{\bar{I}_y - \bar{I}_z}{\bar{I}_x} \quad \text{and} \quad k_2 = \frac{\bar{I}_z - \bar{I}_x}{\bar{I}_y} \tag{61}$$

which totally specify the moments of inertia of the tip mass if we assume it has a fixed mass m_2 and a constant density ρ. The rest of the parameters are taken as follows:

$$\omega_x(0) = 0.1 \text{ rad/s}, \quad \omega_z(0) = 0.15 \text{ rad/s}, \quad \omega_y(0) = 0 \text{ rad/s}$$
$$\omega_s(0) = 50 \text{ rad/s}, \quad \alpha(0) = 0.1 \text{ rad}, \quad \dot{\alpha}(0) = 0 \text{ rad/s}$$
$$A = 600 \text{ kg m}^2, \quad C = 1000 \text{ kg m}^2, \quad I_r = 200 \text{ kg m}^2$$
$$m_2 = 20 \text{ kg}, \quad K_y = 523 \text{ kg m}^2/\text{s}^2, \quad L = 1.5 \text{ m}$$
$$\rho = 100 \text{ kg/m}^3, \quad \epsilon = 0.01 \tag{62}$$

A graph of the perturbation magnitude of $\bar{\omega}_x$ determined by the root mean square of the coefficients vs k_1 and k_2 is shown in Fig. 3. The region of large perturbation corresponds to a resonance between the torsional vibration of the beam and the precession of the satellite as a whole and actually extends much higher than shown on the graph. This is not unexpected and could easily be found from a simple analysis of the problem. However, as shown in Fig. 4, this graph also depicts what is happening away from resonance, information that could not be predicted by a first examination of the problem. Figure 4 is a cutaway of Fig. 3 along the line $k_1 = 0$. Two facts are now apparent that a simple resonance study would not have revealed. There is some width to the peak at the resonance, so that even if the system is not exactly at resonance, there may be some problems. There also is a local minimum in the perturbation amplitude around $k_2 = -0.25$. Although not a great deal lower than the surrounding values, if $\bar{\omega}_x$ is a variable that is to be closely controlled, it makes sense to design the appendage with the shape that minimizes the perturbation of $\bar{\omega}_x$. The values of the rest of the parameters for $k_1 = 0$ and $k_2 = -0.25$ are

$$I_x = 26.5 \text{ kg m}^2, \quad I_y = 21.2 \text{ kg m}^2, \quad I_z = 21.2 \text{ kg m}^2 \tag{63}$$

We now compare the perturbation solution for these values of the parameters with a numerical integration of the *full* equations of motion, Eqs. (7-11), which we will treat as the "exact" motion that the satellite undergoes. Figure 5 shows the amplitude of the value of $\bar{\omega}_x$ as a function of τ. Figure 6 shows the amplitude of $\bar{\omega}_y$, Fig. 7 shows the amplitude of $\bar{\omega}_z$, and Fig. 8 shows the amplitude of $\bar{\alpha}$, all as functions of τ.

In all of the graphs, both the numerically integrated solution that is nondimensionalized and the perturbation solution that is already nondimensional are plotted on the same graph. Figures 5, 6 and 8 all apparently show that the perturbation solution gives an exact fit. This is not the case as can be seen in Fig. 7, which shows that the perturbation solution actually slightly overestimates the variation from the rigid-body solution. This figure is of greatly different scale than those for the other variables since $\bar{\omega}_z$ is a constant in the unperturbed case, and hence its graph is centered at that unperturbed value and not centered at zero as is the case for the other figures. However, it is clear that the perturbation solution gives a very good approximation to the numerically integrated full equations of motion, and therefore it would seem to be a good indicator of the expected behavior of the system at a point away from resonance.

Conclusion

From the results presented in this paper, it is apparent that the KBM method as applied here has promise as an easy way to analyze the motion of spacecraft undergoing a small perturbation due to flexibility. Once the equations are found for a displacement mode, the effect of any given parameter may be sought without resorting to repeated applications of a numerical integrator. Although some resonances are found, the main strength of the approach is the ability to locate local minima of the perturbations in parameter space, a design tool that might help reduce the size of the control effort needed for accurate pointing of the spacecraft. In addition it is shown that the perturbation solution compares quite variably with the exact solution obtained by numerical integration, in both frequency and amplitude. It is much easier to compute as well, since it is composed of the sum of sines and cosines instead of a numerical integration of a complicated set of equations. This is important if prediction of motion is desired in addition to simply finding the total size of the perturbation.

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